NASA TECHNICAL TRANSLATION NASA TT F-14443

TO THE THEORY ON RADIATION TRANSFER IN A NON-HOMOGENEOUS MEDIUM

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(NASA-TT-F-14443) TO THE THEORY ON RADIATION TRANSFER IN A NONHOMOGENEOUS MEDIUM V.V. Zheleznyakov (NASA) Jun. 1972 16 p CSCL 20M N72-32947 G3/33 Unclas 43362

Translation of an article in Izvestiya vysshikh uchebnykh zavedenii (Radiofizika), Vol. 9, No. 6, 1966, pp. 1057-1064.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546 JUNE 1972

Item 27

Source: To the Theory on Radiation Transfer in a Non-Homogeneous Medium

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The well-known equation on the radiation transfer in a non homogeneous medium (1) is derived from the continuity (conservation) equation for the density of radiation energy in the space co-ordinates and in the directions of group velocity. The Oster theory of transfer in non-homogeneous medium (1) is shown to be inaccurate; it was, therefore, erroneously applied by Oster and Sofia (2) in their research on radio emissions of the quiet Sun.

1. The problem of radiation transfer plays a basic part in many problems of astro-physics and radio-astronomy, thus justifying the publication of a special article to establish a true transfer equation. We will demonstrate below the inaccuracy of the radiation transfer equation in a non-homogeneous medium as established by Oster ([1]) and the correctness of the same equation in form (1). We will then select an expression for the radiation capacity μ in the articles ([1,2]) and discuss the results derived when applying Oster radiation transfer theory to the problem of plasma radiation, and in particular, to the problem of radiowaves from the quiet Sun.

It is a well-known fact (see, for instance ([3,4]) and ([5]), paragraph 26) that the equation of radiation transfer in a non-homogeneous isotropic medium is as follows:*

$$n^2 \frac{d}{dl}\left(\frac{I}{n^2}\right) = a - \mu I. \qquad (1)$$

*At this time, as well as below, the spectral values of intensity I, the radiation capacity a, etc. are being considered. However, in order to simplify the recording, we disregard, as a rule, in this case, the index ω in the corresponding values.

where n is the refraction index, I-the radiation intensity in relation to the single solid angle a is the radiation capacity of the unit volume, μ the absorption coefficient, dl the element of ray length. This equation is correct in an area where the geometrical optics have a regular approximation and is recorded for a stationary case when I does not depend upon the time t.

The equation (1) which determines the balance of radiation energy along the ray is usually obtained in the following manner. If the transparent $(\mu = 0)$ and the non-radiating $(a = 0)$ medium is non-homogeneous, the radiation intensity, along the ray changes proportionally to n^2 due to refraction. In other words,

$$\frac{I}{n^2} = \text{const}, \quad \frac{d}{dl}\left(\frac{I}{n^2}\right) = 0 \qquad (2)$$

or,

$$\frac{dI}{dl} = \frac{2I}{n}\frac{dn}{dl} \qquad (3)$$

which is the same.

(see, for instance, (3) and (5), para 26). The effect of absorption $(\mu \neq 0)$ and radiation $(a \neq 0)$ in the medium will reveal itself in the additional change in intensity along the ray; this change corresponds to $a - \mu I$ per one unit of the ray length. By adding the last value to the change I by means of refraction (3) we obtain the relation

$$\frac{dI}{dl} = \frac{2I}{n}\frac{dn}{dl} + a - \mu I,$$

which coincides with the equation on radiation transfer in a non-homogeneous isotropic medium (1)*.

These considerations seem to be sufficiently convincing. Yet doubts as to the accuracy of equation (1) were expressed in the references ([1],[2]) as well as in the article ([3]).

In the article ([1]) Oster suggested a different equation for the radiation transfer in a non-homogeneous medium, which is presented below:

$$v_{rp}\nabla f = q - \sigma f. \qquad (4)$$

where v_{rp} is the photon group velocity, f - the number of photons in a unit of volume and solid angle, q - the number of emitted photons per volume unit and in a unit of time and solid angle, σ when multiplied by f, is a factor characterising the number of absorbed photons. By multiplying both components of the equation (4) by the photon energy we reduce (4) to:

$$v_{rp}\nabla u = a - \sigma u, \qquad (5)$$

where the photon density f is now substituted by the usual macroscopic value of the radiation energy density in a single solid angle $u = \hbar\omega f.$ Inasmuch as the ray direction coincides with the group velocity

*A generalization of the radiation transfer equation in case of a non-homogeneous anisotropic medium is presented in ([5]), para 26:

$$(n_j^2/|\cos\vartheta|)d(I_j|\cos\vartheta|/n_j^2)/dl = a_j - \mu_j I_j.$$

Here the ϑ is the angle between the wave vector k_j and the group velocity $d\omega/dk_j$; the index j shows that the corresponding value relates to a wave of one type (regular or irregular).

direction, the equations (4), (5) may be presented as:

$$v_{\text{rp}} \frac{df}{dl} = q - \sigma f, \tag{4a}$$

$$v_{\text{rp}} \frac{du}{dl} = a - \sigma u. \tag{5a}$$

In connection with Oster equations the following could be said. Equation (4) for photons in a medium has been written by Oster by analogy with the kinetic equation for particles in the absence of forces:

$$\frac{\partial f}{\partial t} + v\nabla_r f = G, \tag{6}$$

where f is the particle distribution function, G - the component which takes into account the "birth" and the "disappearance" of particles. In a stationary case $(\partial f/\partial t = 0)$ and providing $G \equiv q - \sigma f$, this equation is transmuted into (4).

The desire to compose a radiation transfer equation by the same method as the kinetic equation for particles is quite justified. Actually, the kinetic equation composed as

$$\frac{\partial f}{\partial t} + v\nabla_r f + \frac{F}{m}\nabla_v f = G \tag{7}$$

(where F - is the force, acting on the particle by the mass m), is a result of the continuity equation (conservation equation) of particles density in a phase space of co-ordinates r and velocities v*:

$$\frac{\partial f}{\partial t} + \nabla_r(\dot{r}f) + \nabla_v(\dot{v}f) = G \tag{8}$$

*Periods here mark total derivatives in time t.

and the particle motion equations:

$$\dot{r} = v; \qquad \dot{v} = F(r)/m. \tag{9}$$

An analogous correlation to (8) exists for the energy radiation density u when relating to a single solid angle. However, during the process of deriving the transfer equation from the continuity equation (conservation equation) we arrive at the relation (12a) which does not coincide with the Oster equation (5); yet the transfer equation (1) can be derived from this relation - which we will readily demonstrate.

2. Thus, we first derive the equation of radiation transfer in a non-homogeneous medium from the continuity equation for radiation density u.

First, it should be taken into account that the derivative r which now represents the transfer velocity of the radiation energy coincides with the group velocity $v_{rp} = d\omega/dk$ (k is the wave vector). The absolute value of the group velocity

$$v_{rp} = \left|\frac{d\omega}{dk}\right| = c\left[\frac{\partial(\omega n)}{\partial\omega}\right]^{-1} \tag{10}$$

is unabiguously connected with the value of the radius-vector r (through the refraction index $n = n(\omega, r)$). The energy density u is therefore (to the contrary of function f (r, v, t) for particles) only a function of the time, the radius-vector r and the direction of the group velocity l:

$$u = u(r, l, t) \tag{11}$$

(l is a single vector directed on a tangent to the ray, in other words, along the group velocity: $v_{rp} = l v_{rp}$). It becomes clear now that the phase space of the radiation will be $r, l,$ unlike the case of the $r, v;$ of the material particles; accordingly, the continuity equation for radiation density in space r, l will be recorded in the following manner: *

$$\frac{\partial u}{\partial t} + \nabla_r(v_{rp}u) + \nabla_l(\dot{l}u) = G \equiv a - \sigma u \qquad (12)$$

($\nabla_l(\dot{l}u)$ stands for a two-dimensional divergence on a single vector l). The second equation (9) is, evidently, not applicable to radiation; it should be substituted by an equation defining the ray configuration in geometrical optics approximation (See ([6]), para 65):

$$\frac{dl}{dl} = \frac{1}{n}\left[\nabla_r n - l(l\nabla_r n)\right]. \qquad (13)$$

The relation (12) does not correlate with Oster equation (5). It is true that when we transcribe the relation (12) as follows:

$$\frac{\partial u}{\partial t} + v_{rp}\nabla_r u + u\nabla_r v_{rp} + \nabla_l(\dot{l}u) = a - \sigma u, \qquad (12a)$$

*By integrating the equation by terms (12) along all directions l (along all solid angles $d\Omega,$)and taking into account that $\int_{4\pi} u\,d\Omega = w$ are the radiating energy densities $\int_{4\pi} v_{rp} u\,d\Omega = s$ s - is the Pointing vector $\int_{4\pi} \nabla_l(\dot{l}u)\,d\Omega = 0,$ we find, G = 0, the Pointing theorem in a non-absorbing medium:

$$\partial w/\partial t + \nabla_r s = 0.$$

we perceive that in a stationary case $(\partial u/\partial t = 0)$ it differs from (5) in the terms $u\nabla_r v_{rp}$ и $\nabla_l(Iu)$. On the other hand, it is not difficult to show that, taking into consideration (13) and under conditions of $\partial u/\partial t = 0$ the equation (12) becomes the well-known radiation transfer equation (1).

In order to prove the last statement we will notice that*

$$
\begin{aligned}
&I = v_{rp}u; \qquad v_{rp}u = lv_{rp}u = lI; \\
&\dot{l} = \frac{dl}{dl}\,\frac{dl}{dt} = \frac{dl}{dl}v_{rp}; \quad \sigma = \mu v_{rp}.
\end{aligned}
\tag{14}
$$

Taking into account these evident relations, we transcribe (12) in the form

$$
\frac{\partial u}{\partial t} + \nabla_r(lI) + \nabla_l\left(\frac{dl}{dl}\,I\right) = a - \mu I \tag{15}
$$

or

$$
\frac{1}{v_{rp}}\,\frac{\partial I}{\partial t} + l\Delta_r I + I\nabla_l\frac{dl}{dl} + \frac{dl}{dl}\,\nabla_l I = a - \mu I. \tag{16}
$$

When passing over to (16) the equality $u = I/v_{rp}$, was taken into account; the v_{rp} depends only upon r, and does not depend upon t (in a medium with properties not changing with time); in addition the fact that $\nabla_r l = 0,$ is taken into consideration inasmuch as vector l

*The equality $I = v_{rp}u$ is true only in a weak absorption medium. The latter serves, together with the requirement of validity of the geometric-optical approximation as a condition for the application of the transfer equation.

is an independent variable together with r. By substituting the expression for dl/dl (13) in $\nabla_l(dl/dl)$, we obtain

$$\nabla_l \frac{dl}{dl} = -\frac{1}{n}(l\nabla_r n)\,\nabla_l l - l\nabla_l\left\{\frac{1}{n}(l\nabla_r n)\right\}.$$

However, $\nabla_l l = 2$; in addition $l\nabla_l\left\{\frac{1}{n}(l\nabla_r n)\right\} = 0$, inasmuch as the gradient on the single vector l is orthogonal to this vector. Therefore,

$$\nabla_l \frac{dl}{dl} = -\frac{2}{n}\,(l\nabla_r n)$$

and, consequently, equation (16) passes into

$$\frac{1}{v_{\mathrm{rp}}}\frac{\partial I}{\partial t} + l\nabla_r I - I\frac{2}{n}\,l\nabla_r n + \frac{dl}{dl}\,\nabla_l I = a - \mu I.$$

By dividing all terms of the equation by n^2 and somewhat transforming it, we get

$$\frac{1}{n^2 v_{\mathrm{rp}}}\,\frac{\partial I}{\partial t} + l\nabla_r\left(\frac{I}{n^2}\right) + \frac{dl}{dl}\nabla_l\left(\frac{I}{n^2}\right) = \frac{a}{n^2} - \mu\,\frac{I}{n^2}. \qquad (17)$$

The relation (17) represents an equation with partial derivatives for I with independent variables r, l, t.

Let us consider, at present, the changes in relation I/n^2 along a definite ray:

$$r = r(l); \qquad l = l(l), \qquad (18)$$

where l is the coordinate along the ray (the length of the ray). Inasmuch as both the r and the l change when l changes, the derivative of I/n^2 along the ray is

$$\frac{d}{dl}\left(\frac{I}{n^2}\right)=\frac{dr}{dl}\nabla_r\left(\frac{I}{n^2}\right)+\frac{dl}{dl}\nabla_l\left(\frac{I}{n^2}\right)=l\nabla_r\left(\frac{I}{n^2}\right)+\frac{dl}{dl}\nabla_l\left(\frac{I}{n^2}\right). \quad (19)$$

(In passing over the last equality it has been taken into account that $dr/dl=l$.) In the right part of the equation (19) vectors r and l are independent - they are interconnected over the parameter (see (18)). The value $l\nabla_r(I/n^2)+(dl/dl)\nabla_l(I/n^2)$ presented in this part may be found from equation (17); it equals:

$$\frac{a}{n^2}-\mu\frac{I}{n^2}-\frac{1}{n^2 v_{rp}}\frac{\partial I}{\partial t} \quad (20)$$

under conditions that the r and l values are computed along the ray (18). Substituting (20) into (19), we obtain a final

$$\frac{1}{v_{rp}}\frac{\partial I}{\partial t}+n^2\frac{d}{dl}\left(\frac{I}{n^2}\right)=a-\mu I \quad (21)$$

or

$$\frac{1}{c}\frac{\partial(\omega n)}{\partial\omega}\frac{\partial I}{\partial t}+n^2\frac{d}{dl}\left(\frac{I}{n^2}\right)=a-\mu I. \quad (21a)$$

which is the same thing.

In a stationary case it coincides with the well-known transfer equation (1). We notice that equation (17) was obtained actually in an analogous manner by Harris (8). However, he was doubtful of the accuracy of the equality (19) which gave him grounds for the statement that equation (17), in the final count, cannot be reduced to equation (21).

3. Let us consider the differences in intensity of radiation which result when the true transfer equation is applied (1) and when Oster equation (5a) is used in a system where particles are in the equilibrium state (in the sense of their velocity distribution). The radiating capacity of such a system a is connected with the absorption coefficient μ of the Kirchoof equation

$$a = \mu I^{(0)}, \tag{22}$$

where $I^{(o)}$ - is the intensity of the equilibrium radiation of this system. (If the refraction index in this system equals n, then

$$I^{(0)} = n^2 I_0^{(0)}, \tag{23}$$

where $I_0^{(0)}$ is the equilibrium intensity in a vacuum; (See (5), para 26). Taking into account (22) and (23), the first and the last from the equality (14) the equation (1), (5a) become

$$\frac{d}{dl}\left(\frac{I}{n^2}\right) + \mu \frac{I}{n^2} = \mu I_0^{(0)}; \tag{24}$$

$$\frac{d}{dl}\left(\frac{I}{v_{rp}}\right) + \mu \frac{I}{v_{rp}} = \mu I_0^{(0)} \frac{n^2}{v_{rp}}. \tag{25}$$

The solution of the transfer equation (24) can be recorded in the form (para 26, (5)).

$$\frac{I}{n^2} = e^{-\tau} \int_0^{\tau} I_0^{(0)} e^{\tau} d\tau + e^{-\tau} \left(\frac{I}{n^2}\right)_{l=l_0}, \tag{26}$$

where the optical density τ is expressed by using the μ integral (along the ray).

$$\tau = \int_{l_0}^{l} \mu \, dl.$$

At the same time the expression

$$\frac{I}{v_{rp}} = e^{-\tau}\int_0^{\tau} I_0^{(0)} \frac{n^2}{v_{rp}} e^{\tau} d\tau + e^{-\tau}\left(\frac{I}{v_{rp}}\right)_{l=l_0} \tag{27}$$

serves as a solution to Oster's equation (25).

In the homogeneous medium $(v_{rp} = \text{const},\ n^2 = \text{const})$ both expressions (26), (27) coincide. However, in a non-homogeneous medium they yield different results. Thus, for instance, the radiation intensity transferring from the absorbing layer into the vacuum equals according to (26)

$$I = e^{-\tau}\int_0^{\tau} I_0^{(0)} e^{\tau} d\tau \tag{28}$$

while (27) in this case becomes:*

$$I = e^{-\tau}\int_0^{\tau} I_0^{(0)} \frac{cn^2}{v_{rp}} e^{\tau} d\tau. \tag{29}$$

The expression (29) differs from the correct equation (28) by the presence of the factor cn^2/v_{rp} in the integrand expression. In plasma $v_{rp} = cn$ and the factor $cn^2/v_{rp} = n < 1$; it becomes clear that the values obtained when the Oster equation is

*In the transition from (27) to (29) the $v_{rp} = c$ in the vacuum is considered to be equalling c.

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applied to the plasma radiation problem are too low (at equal values of τ).

The inaccuracy of the solution (29) is clearly seen in the case of radiation inside a non-transparent cavity with a temperature T. If the refraction index is, in the center of this cavity n = 1, then according to thermodynamics, there exists in the cavity an equilibrium radiation with an intensity of $I = I_0^{(0)}$ regardless of the type of walls - the values n, v_{rp} etc (the only requirement being the non-transparency of the walls.). The same result is obtained from the solution (28): at a T = const the value $I = I_0^{(0)}(1 - e^{-\tau})$ is true for non-transparent walls with a $\tau \gg 1$ the intensity in the cavity is $I = I_0^{(0)}$. Yet, when applying the Oster equation a different result is obtained: at a T = const and a constant value of cn^2/v_{rp} in the walls* the value $I = I_0^{(0)}(cn^2/v_{rp})(1 - e^{-\tau})$; if the walls are non-transparent $(\tau \gg 1)$, then contrary to thermodynamics, the intensity in the cavity** $I = I_0^{(0)}(cn^2/v_{rp}) \neq I_0^{(0)}$.

In addition to an incorrect equation for transfer, the articles (1,2) also use an erroneous expression for the absorption coefficient μ, which determines the optical density $\tau = \int \mu\, dl$

*The transfer equation (1) and its solution (28) are recorded under conditions that the radiation reflection from sharp borders is not present. Therefore, in the example analyzed, the value cn^2/v_{rp} must be constant only in those layers where the absorption coefficient is $\mu \neq 0$. To achieve an absence of reflections it is necessary to have these layers separated by a smoothly-non-homogeneous medium from the center of the cavity.

**The fact that Oster transfer equation disagrees with thermodynamics was mentioned in Cronyn's article at an earlier date.

Oster contends that the radiation capacity of the system $a = h\ldots$ does not depend upon the presence of the medium, in other words, upon the refraction index n. In accordance with the Kirchoof law $a = \mu I^{(0)}$ t thereby follows that $\mu \backsim n^{-2}$ (providing the equilibrium intensity is $I^{(0)} \backsim n^2$ see paragraph 26 (5)). In other words, the absorption coefficient in the medium is

$$\mu = \mu_0 n^{-2}, \tag{30}$$

where μ_0 is the absorption coefficient of the system of radiating particles in a vacuum. According to Oster, the, it is sufficient to know the absorption coefficient in a vacuum to be able to determine the absorption coefficient in a medium.

Yet, it is a well known fact that the presence of a medium with a refraction index $n \neq 1$ substantially changes the character of radiation of individual elemental centers (charged particles, atoms, molecules). Thus, the bremstrahlung intensity of one electron is proportional to n (in the dipole approximation); at sufficiently large n values of electrons moving at a v velocity, the Vavilov-Cherenkov radiation is generated which is absent in a vacuum, etc. It is clear that the radiation characteristics of a system of such particles change considerably depending upon the presence or the absence of a medium, and, in each case, this change will be different.

For example, we will analyse the breamstrahlung in the connection with plasma absorption. The radiation capacity is a proportionate to n (as in the case of an individual electron; see above); the value

$$\mu = \frac{a}{I^{(0)}} \backsim n^{-1}, \qquad (31)$$

while according to Oster it is $\mu \backsim n^{-2}$. The accuracy of the expression (31) is proven also by the expression for the absorption coefficient of electromagnetic waves in isotropic plasma derived from the elementary theory (See para 7, (7)):

$$\mu = \frac{2\pi e^2 N \nu_{\text{эфф}}}{m\omega(\omega^2 + \nu^2_{\text{эфф}})n}. \qquad (32)$$

Equation (32) characterizes the absorption connected with the bremstrahlung by the system of N electrons which undergo collisions with ions with a frequency $\nu_{\text{эфф}}$; it is also inversely proportional to n. (In (32) e is the electron charge).

It follows from the above that the transfer equation in a non-homogeneous medium introduced by Oster and his method of computing the absorption coefficient in a medium by means of the value of the absorption coefficient at $n = 1$ are incorrect. For these reasons the results of the extensive computations of radio-luminosity on the disk of the "quiet" Sun performed by Oster and Sofia in article (2) are incorrect for two reasons. First, the corona and the chromosphere are non-homogeneous; as explained above, in such cases, the use of the transfer equation in the form (5), (5a), results in substantial errors, by underestimating the value of the radio emission intensity. By comparing equations (28), (29) it is easy to notice that in different points of the disk these errors will vary; therefore, not only the absolute value of intensity, but the character of distribution of radio luminosity on the disk will differ from the true values. In the second place, a computation of the absorption coefficient of the optical density of the sun corona and chromosphere by means of a "universal" relation between the absorption coefficients in the medium and in a vacuum (30) also leads to incorrect results.

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